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Technical note

Investigation on power output of the gamma-configuration low temperature differential Stirling engines

Bancha Kongtragool, Somchai Wongwises *

Fluid Mechanics, Thermal Engineering and Multiphase Flow Research Laboratory (FUTURE), Department of Mechanical Engineering, King Mongkut's University of Technology Thonburi, 91 Suksawas Road, 48, Thungkru, Bangmod, Bangkok 10140, Thailand

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Abstract

This paper provides a study on power output determination of a gamma-configuration, low temperature differential Stirling engine. The former works on the calculation of Stirling engine power output are discussed. Results from this study indicate that the mean pressure power formula is most appropriate for the calculation of a gamma-configuration, low temperature differential Stirling engine power output.
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Keywords: Stirling engine; Hot-air engine

1. Introduction

Stirling engines are mechanical devices working theoretically on the Stirling cycle, or its modifications, in which compressible fluids, such as air, hydrogen, helium, nitrogen or even vapors, are used as working fluids. The Stirling engine offers the possibility of a high efficiency engine with lower exhaust emissions in comparison with the internal combustion engine. In principle, the Stirling engine is simple in design and construction, and can be operated easily.

The earlier Stirling engines were highly inefficient. However, over time, a number of new Stirling engine models have been developed to improve the initial

* Corresponding author. Tel.: +66-2-470-9115; fax: +66-2-470-9111.
E-mail address: somchai.won@kmutt.ac.th (S. Wongwises).

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Nomenclature

e	regenerator effectiveness
E_{BT}	brake thermal efficiency
E_H	heat source efficiency
E_M	mechanical efficiency
E_O	overall Stirling engine powered system efficiency
E_S	thermodynamic efficiency of the Stirling engine
f	engine speed or cycle frequency, rps or Hz or s^{-1}
F_E	the empirical factor for the engine only
F_O	overall empirical factor
k	specific heat ratio
k_P	V_P/V_D is swept volume ratio
k_S	V_S/V_D is dead space volume ratio
K_C	the Stirling coefficient
N_B	Beale number, W/(bar cm^3 s^{-1})
p_m	mean or average cycle pressure, N/m^2, bar
p_{max}	maximum pressure attained during cycle, N/m^2
P	engine shaft power output, W
P_I	engine indicated power, W
q_{in}	heat input or useful energy, W
T_H	the hot space working fluid temperature, K
T_C	the cold space working fluid temperature, K
V_1	$V_D + V_C$ is volume at state 1, m^3
V_2	V_D is volume at state 2, m^3
V_D	displacer swept volume, m^3
V_P	power piston swept volume, m^3, cm^3
V_S	dead space volume, m^3
W_{net}	net indicated work per cycle, N m
$W_{Schmidt}$	indicated work per cycle calculated from Eq. (6), Nm
W_{West}	indicated work per cycle calculated from Eq. (10), Nm

Greek letters

α	phase angle lead of the displacer over the power piston, degrees.
τ	T_C/T_H is working fluid temperature ratio
τ'	sink to source temperature ratio

deficiencies. The modern Stirling engine is more efficient than the early engines and can use any high temperature source. The Stirling engine is an external combustion engine. Therefore, most sources of heat can power it, including combustion of any combustible material, field waste, rice husk or the like, biomass methane and solar energy.

When a new engine is in the initial design phase, the design performance evaluations needed may be that of simple calculations required to ascertain the possible performance of the engine. It is important that the rated power output, at a chosen design speed, should be specified.

Appropriate performance criteria are also required in order to determine the performance of a Stirling engine. If a full assessment of the proposed engine is to be determined, more detailed data will be required. However, in the initial design phase many design parameters are unknown and we therefore need to know the possible engine power output calculated from a simple formula. To do this, we should use an accurate formula that involves the least number of design parameters. There are many methods that can be used to obtain design power output.

The Malmo formula [1] could be the simplest formula for the calculation of Stirling engine power output. The Malmo formula requires only the heat-input value from the heat source and the empirical factors. However, the results from Malmo formula are mostly dependent on empirical factors with values varying in a wide range.

For Stirling engine output calculation, the formula by Schmidt [2] gives a mathematically exact expression of the indicated work per cycle satisfying the isothermal and sinusoidal motion assumptions. Reader and Hooper [1] presented a comprehensive derivation of the Schmidt formula for the alpha-, beta- and gamma-configuration Stirling engine. Although Martini [3] gave the Schmidt formula for indicated work in a fairly simple form, West [4] proposed a simpler formula. An approximate formula to calculate the Stirling engine shaft output can be obtained using the Schmidt or West formula with an appropriate experience factor.

The Schmidt and West formula contains several parameters that would not necessarily be known in the initial design phase. The simpler approach is the use of the Beale formula. Beale observed that the power output of many Stirling engines could be expressed by an empirical formula [1,5]. This empirical formula is very simple and originated the Beale number concept. The Beale formula only requires a few design parameters and is very convenient in the preliminary design phase.

Walker [6] presented the Beale number in a non-dimensional form that gave rise to the Beale number concept. West [7] used this concept in the design of the Fluidyne engine. Jones and Hua [8] suggested the Beale number as a fuzzy number in the routine engineering design of Stirling engines.

The generalized treatment of the Beale numbers correlation, which was later developed by Walker [6], West [9], and Senft [10], gives a mean pressure power formula as a useful design guideline. Audy et al. [11] used this formula to calculate the power output of real engines with losses.

Although many researchers have studied the Stirling engine power output determination (see Kongtragool and Wongwises [12]), there still remains room for Stirling engines operating at low temperatures. There are some existing data for the Beale number used with the Beale formula, but these data likely unusable for a Stirling engine operated at relatively low temperature. The low temperature differential (LTD) Stirling engine is one that operates at comparatively low temperature and should be studied in detail.

The objective of this article is to provide a basic background and review of existing literature on calculation of LTD Stirling engine power output. A number of Stirling engine power output formulas are provided and discussed. The aim of this article is to find out a simple and sufficiently accurate formula to calculate the power output of a gamma-configuration LTD Stirling engine.

2. Engine power output from Malmo formula

A simple formula for Stirling engine power output could use the concept of brake engine efficiency

$$P = E_{BT} q_{in} \tag{1}$$

where P is shaft power output in W, E_{BT} is brake thermal efficiency, and q_{in} is heat input or useful energy in W. A similar formula for Stirling engine system efficiency was first presented in a United Stirling publication, called the Malmo formula [1]. The generalized Malmo formula can be expressed in a simple form

$$E_O = F_O E_S \tag{2}$$

where E_O is overall Stirling engine powered system efficiency, F_O is overall empirical factor and E_S is thermodynamic efficiency of the Stirling engine. The value of F_O will be in the range of about 0.3–0.68 [1]. When only the Stirling engine is considered, Eq. (2) is then changed to

$$E_{BT} = F_E E_S = (E_H E_M K_C) E_S \tag{3}$$

where F_E is the empirical factor for the engine only, E_H is heat source efficiency, E_M is mechanical efficiency, K_C is the Stirling coefficient. The value of F_E would be in the range of about 0.35–0.75. The values of these empirical factors are shown in Table 1. The engine shaft power is therefore:

$$P = F_E E_S q_{in} = (E_H E_M K_C) E_S q_{in} \tag{4}$$

It is evidenced that the Malmo formula seems to be the simplest formula for the calculation of Stirling engine power output. This formula requires only the empirical factors, heat input from the heat source and the Stirling cycle efficiency. However, the results from Malmo formula are mostly dependent on empirical factors with values varying in a wide range.

Table 1
Values of empirical factors in Malmo formula

Factor	Range of values
E_H	0.85–0.95
E_M	0.75–0.90
K_C	0.55–0.88

Source: Reader and Hooper [1].

In general, Stirling cycle efficiency depends on the source and sink temperatures and the regenerator effectiveness. Since in the ideal regeneration, the infinite heat-transfer area or the infinite regeneration time is needed, then, the analysis with imperfect regeneration should be made. Although the effectiveness of 95%, 99.09% and 98–99% are reported [13–17], the engine developers who do not have in hand the efficient-regenerator technology should be taken into account the regenerator effectiveness. The Stirling engine cycle efficiency can be determined from [18]:

$$E_S = (1-\tau)/\{1+(1-e)(1-\tau)/[(k-1)(\ln V_1/V_2)]\} \quad (5)$$

where $\tau = T_C/T_H$ is working fluid temperature ratio, T_H is the hot space working fluid temperature in K, T_C is the cold space working fluid temperature in K, e is the regenerator effectiveness, k is specific heat ratio, $V_1 = V_D + V_C$ is volume at state 1 and $V_2 = V_D$ is volume at state 2 of the compression process, V_D is displacer swept volume in m^3 and V_P is power piston swept volume in m^3.

Therefore, the Malmo formula actually requires five design parameters, namely displacer and power piston swept volumes, the regenerator effectiveness, the heater and the cooler temperatures of the engine, together with the heat input from the heat source and three empirical factors.

3. Engine power output from indicated work

Schematic diagram of a gamma-configuration Stirling engine and sinusoidal motion diagram of displacer and power piston are shown in Fig. 1. According to the mechanical configurations of Stirling engines, which are classified into the alpha-, beta-, and gamma-configuration, slight differences in their p–v diagrams result in the cycle net works not being the same. The Schmidt or West formula can be used to make a calculation of engine shaft power.

3.1. Schmidt formula

Schmidt [2] showed a mathematically exact expression for determining the indicated work per cycle of a Stirling engine. The Schmidt formula may be shown in various forms depending on the notations used. The formula for gamma-configuration Stirling engines is as follows [19]:

$$W_{Schmidt} = p_m V_P \pi \sin\alpha (1-\tau)/[A + (A^2 - B^2)^{1/2}] \quad (6)$$

and

$$A = (1+\tau) + (4k_S\tau)/(1+\tau) + k_P \quad (7)$$

$$B = [(1-\tau)^2 - 2(1-\tau)k_P \cos\alpha + k_P^2]^{1/2} \quad (8)$$

$$p_m = p_{max}[(A-B)/(A+B)]^{1/2} \quad (9)$$

where $W_{Schmidt}$ is indicated work per cycle in Nm, p_m is mean or average cycle pressure in N/m^2, p_{max} is maximum pressure attained during cycle in N/m^2, $k_P = V_P/V_D$ is swept volume ratio, $k_S = V_S/V_D$ is dead space volume ratio, V_S is dead

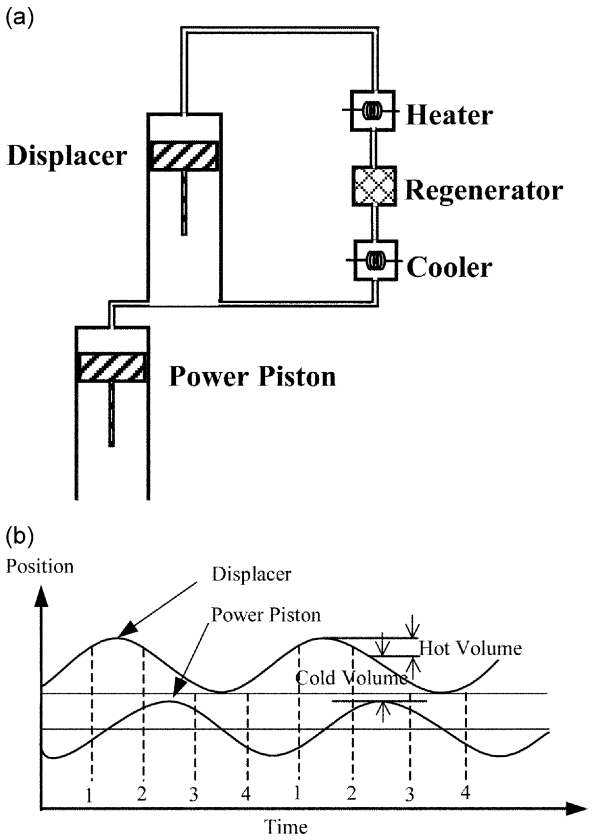

Fig. 1. (a) Schematic diagram of a gamma-configuration Stirling engine and (b) sinusoidal motion diagram of displacer and power piston.

space volume in m^3, and α is phase angle lead of the displacer over the power piston in degrees.

3.2. West formula

West [19] proposed a simpler formula to determine the indicated work as follows [19]:

$$W_{\mathrm{West}} = p_{\mathrm{m}} V_{\mathrm{P}} \pi \sin\alpha [(1-\tau)/(1+\tau)]/[2 + k_{\mathrm{P}} + 2k_{\mathrm{S}}] \tag{10}$$

Eq. (10) gives an error for the indicated work for sinusoidal motion compared to the exact solution from Eq. (6).

3.3. Indicated power

The indicated power can be calculated from

$$P_{\mathrm{I}} = W_{\mathrm{net}} f \tag{11}$$

where P_I is indicated power in W, W_{net} is net indicated work per cycle in Nm and f is cycle frequency in s^{-1} or engine speed in revolutions per second. The work per cycle can be either $W_{Schmidt}$ or W_{West}.

3.4. Shaft power

The engine shaft power can be calculated from

$$P = (\text{Experience factor})P_I \tag{12}$$

where P is engine shaft output in W.

Martini [3] recommended that the shaft power be obtained by reducing the Schmidt formula by an "experience factor" of around 35%. Reader and Hooper [1] suggested a factor of 30–50% for a well-designed engine. Walker [5] also suggested that the expected power and efficiency of Stirling engines should be in the range of 30–60% of the Schmidt analysis.

The Schmidt and West formula requires the same seven design parameters, i.e. p_m, V_P, V_D, V_S, T_H, T_C and α, together with an experience factor. Therefore, the Schmidt and West formula are the same degree of complexity.

4. Engine power output from Beale number concept

4.1. Beale formula

Beale [1] noted that the power output of several observed Stirling engines might be calculated approximately from the equation [1]

$$P = (\text{empirical constant})p_m V_p f/\tau' \tag{13}$$

where P is engine power output in W, p_m is mean cycle pressure in bar, V_P is displacement of power piston in cm^3, f is engine speed or cycle frequency in rps or Hz, and τ' is the sink to source temperature ratio. The Beale formula can be used for all configurations and for various sizes of Stirling engines. Eq. (13) can be rearranged as:

$$P/(p_m V_P f) = (\text{empirical constant})/\tau' = N_B \tag{14}$$

The group of parameters $P/(p_m V_P f)$ is called the Beale number, N_B.

When the empirical constant in Eq. (14) has a value of 0.005 with an average source temperature of 923 K and sink temperature of 323 K [1], then $N_B = 0.014\,288$. It is clear that the Beale number is a function of both source and sink temperatures. Eq. (14) may be written as:

$$P = N_B p_m V_p f \tag{15}$$

Eq. (15) is called the Beale formula.

Walker [5] proposed an approximated Beale formula for power output

$$P = 0.015 p_m V_P f \tag{16}$$

or $N_B = 0.015$. Eq. (16) is approximately correct for all types and sizes of Stirling engines.

Therefore, in general, the Stirling engine rated power output can be calculated from the simple Beale formula Eq. (15). The Beale formula requires only the power piston displacement volume, the engine speed and the cycle mean pressure. However, the Beale number is needed in using Eq. (15). The Beale number can be found in many ways.

The first method, the simplest way to obtain the Beale number, is by using the curves provided by Walker [5]. Walker [5] presented the relationship of the Beale number to the source temperature by the full line as shown in Fig. 2. The upper bound represents high efficiency well-designed engines with low sink temperatures, while the lower bound represents moderate efficiency less well-designed engines with high sink temperatures.

As described in Walker [5], in most instances the engines operate with a source temperature of 650 °C and sink temperature of 65 °C. This gives $N_B = 0.015$, approximately, from the full line curve of Fig. 2. However, this method is probably unusable for the engines that have heater temperatures below 475 K, that is the case of LTD Stirling engines, since the curves of Beale number seem to approach zero.

In the second method, the Beale number can be calculated from the second term of Eq. (14). By using the empirical constant = 0.005 [1] in Eq. (14), the Beale number can be calculated from:

$$N_B = 0.005/\tau' \tag{17}$$

The relationship between the Beale number calculated from Eq. (17) and temperature ratio is shown by a dashed line in Fig. 3 for the temperature range of

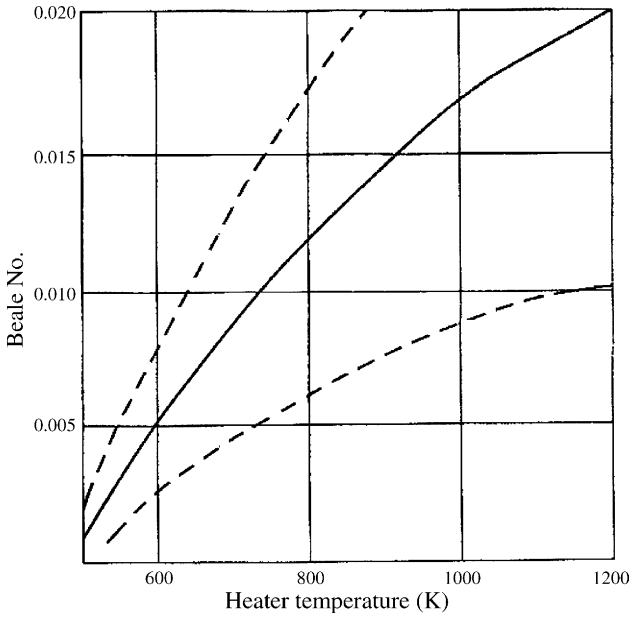

Fig. 2. Beale number as a function of source temperature. *Source*: Walker [5].

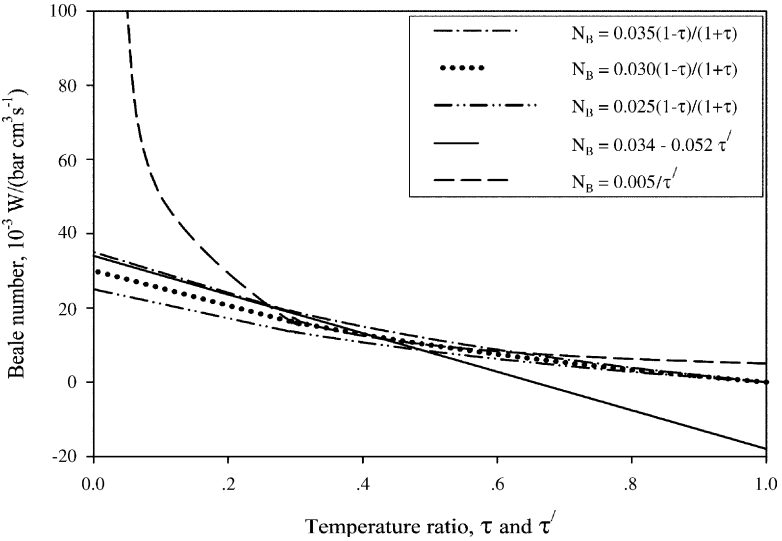

Fig. 3. Beale number calculated from Eqs. (17), (18) and (22).

338–1200 K. It is evidenced that for small temperature ratios, the Beale number approaches infinity. However, for a source temperature equal to sink temperature, $\tau' = 1$, $N_B = 0.005$, so this equation should be used carefully.

The third method, Walker proposed the approximate form of the Beale number as [1]:

$$N_B = 0.034 - 0.052\tau' \tag{18}$$

The relationship between the Beale number calculated from Eq. (18) and the temperature ratio is also shown in Fig. 3 by a solid line. Eq. (17) seems unusable for temperature ratios higher than 0.6539, since beyond this value the Beale number is negative—at this point the source temperature is around 471 K where the sink temperature is 308 K.

4.2. Mean pressure power formula

Walker [6], West [9], and Senft [10] developed the Beale number correlation. Their correlation is used to determine the Stirling engine shaft power output as follows [19]:

$$P = p_m V_P f F[(1 - \tau/(1 + \tau)] \tag{19}$$

where P is engine power output in W, p_m is mean cycle pressure in N/m^2, f is engine speed or cycle frequency in rps or Hz, and V_P is displacement of power piston in m^3. Senft [19] proved that the factor F in Eq. (19) is 2 for the ideal Stirling cycle. However, in this ideal cycle, F does not take into account the mechanical loss, friction etc. Senft [19] and West [4] described that an F value of 0.25–0.35 may

be used in practical use. Senft recommended the higher value of factor F for LTD Stirling engine [19: p. 94].

Eq. (19) can be written in the form:

$$P/(p_m V_P f) = F[(1-\tau)/(1+\tau)] \tag{20}$$

For the units of p_m and V_P in bar and cm^3, respectively, Eq. (20) will be

$$P/(p_m f V_P) = (F/10)[(1-\tau)/(1+\tau)] \tag{21}$$

this is equivalent to the Beale number. Since $F = 0.25$–0.35, then:

$$N_B = (0.025\text{–}0.035)\frac{1-\tau}{1+\tau} \tag{22}$$

Therefore, the fourth method to determine the Beale number is by using Eq. (22). However, the accuracy of Eq. (22) mostly depends on the value of the factor F. The correct value of factor F is critical to an accurate approximation of the power output. The calculated Beale number from Eq. (22) is shown in Fig. 3 by a short-dashed, dot-dashed and dot-dot-dashed line. More details on Beale number for LTD Stirling engines have been investigated in the authors' former work [20].

The Beale formula and mean pressure power formula require the same design parameters, p_m, V_P and operating temperatures, Then, the mean pressure power formula having the same simplicity as the Beale formula. Therefore, by directly using the mean pressure power formula, Eq. (22), with an appropriate factor F, the Stirling engine power output can now be calculated for every temperature ratio.

5. Numerical example

The existing gamma-configuration LTD engine specifications and data collected from Ref. [19] are used as a numerical example. The design and operating parameters and actual shaft power are given in Tables 2 and 3.

Results from calculation, by using the mean pressure power formula, are shown in Table 3. From this table, it can be seen that actual power of ANL engine using He is about 17% higher than the calculated power upper value. The actual power

Table 2
Engine specification and data

	Working fluid	T_H (K)	T_C (K)	P (W)	f (Hz)	V_P (cc)	p_m (bar)
ANL* engine	He	363	283	1.65	2.167	150	1
	Air	363	283	0.7	1.667	150	1
Standard displacer	Air	369	279	0.642	1.75	120	1
Regenerative displacer	Air	369	279	1.05	2.063	120	1
L-27 engine	Air	366	307	0.252	4.5	25	1

Source: Ref. [19: p. 13, 110, 118, 122, 124, 126, 135].

* Argonne National Laboratory.

Table 3
Calculated and actual power output

	Working fluid	Actual P	$F = 0.25$	$F = 0.30$	$F = 0.35$
ANL engine	He	1.65	1.006	1.208	1.409
	Air	0.7	0.774	0.929	1.084
Standard displacer	Air	0.642	0.729	0.875	1.021
Regenerative displacer	Air	1.05	0.860	1.032	1.203
L-27 engine	Air	0.252	0.247	0.296	0.345

of ANL engine using air is about 9.6% lower than the lower value of calculated power. The shaft power of ANL engine with regenerative displacer and L-27 engine are in the range of lower and higher value of calculated power. Therefore, with the appropriate F factor, the approximated gamma-configuration Stirling engine power output with acceptable accuracy can be directly determined using the mean pressure power formula.

6. Conclusions

In the preliminary design phase, some design parameters are unknown. The Schmidt formula and West formula are more difficult to use when compared with the Beale formula and the mean pressure formula. In principle, the Beale formula is simpler, however, an accurate value of the Beale number is critical and the existing data on the Beale number are not available for LTD Stirling engines. The mean pressure power formula gives the same simplicity as the Beale formula and could be used for every temperature ratio and should then be used for this purpose.

For design purposes, the mean pressure power formula can be used to calculate the engine rated output, or inversely, to evaluate the approximate operating parameters of the Stirling engine for a required or given power output. The mean pressure power formula allows us to initiate an initial design process rapidly.

For LTD Stirling engines operated by a low temperature source, results from this study indicate that the rated power output of a LTD Stirling engine can be directly calculated from the mean pressure power formula, Eq. (19), by using an appropriate factor, F.

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